Gravitational Instability and Tsallis' Nonextensive Statistics

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Abstract

The concept of Jeans gravitational instability is rediscussed in the framework of the nonextensive statistics proposed by Tsallis. A simple analytical formula generalizing the Jeans criterion is derived by assuming that the unperturbed self-gravitating collisionless gas is described by the q-parametrized class of nonextensive velocity distribution. It is shown that the critical values of wavelenght and mass depend explicitly on the nonextensive q-parameter. The standard Jeans wavelenght derived for a Maxwellian distribution is recovered in the limiting case q=1. For power-law distributions with cutoff, the instability condition is weakneed with the system becoming unstable even for wavelenghts of the disturbance smaller than the standard Jeans lenght λ_J.

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It is widely known that the statistical description of a large variety of physical systems requires some extension of the standard Boltzmann-Gibbs approach. Particular examples are systems endowed with long duration memory, anomalous diffusion, turbulence in pure-electron plasma, self-gravitating systems or more generally systems endowed with long range interactions. Some years ago, Tsallis proposed the following generalization of the Boltzmann-Gibbs (BG) entropy formula for statistical equilibrium [1]

$$S_q = \left(1 - \sum_i p_i^q\right)/(q-1) \quad , \tag{1}$$

where p_i is the probability of the ith microstate and q is a parameter quantifying the degree of nonextensivity (henceforth Boltzmann' constant is set equal one). In the limit $q \to 1$ the celebrated BG extensive formula, namely

$$S = -\sum_i p_i \ln p_i \quad , \tag{2}$$

is recovered. One of the main properties of S_q is its pseudoadditivity. Given a composite system $A + B$, constituted by two subsystems A and B, which are independent in the sense of factorizability of the microstate probabilities, the Tsallis measure verifies $S_q(A+B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B)$. In the limit $q \to 1$, S_q reduces to the standard logaritmic measure, and the usual additivity of the extensive BG statistical mechanics and thermodynamics is also recovered. Thus, if q differs from unity, the entropy becomes nonextensive (superextensive if $q < 1$, and subextensive if $q > 1$), with the Boltzmann factor generalized into a power law. In other words, $|q-1|$ quantifies the lack of extensivity of the system.

This q-entropy along with its associated thermostatistics is nowadays being hailed as the possible basis of a theoretical framework appropriate to deal with nonextensive settings. A growing body of evidence is now suggesting that the q-entropy may provide a convenient frame for the thermostatistical analysis of many physical scenarios and processes mainly for systems endowed with long-range interactions as happens in astrophysics, plasma physics and cosmology [3]. A large portion of the experimental evidence supporting Tsallis proposal involves a non-Maxwellian (power-law) equilibrium distribution function associated with the thermostatistical description of the classical N-body problem. This equilibrium velocity q-distribution may be derived at least from three different methods: (i) Within the nonextensive canonical ensemble, that is, maximizing Tsallis entropy under the constraints imposed by normalization and the energy mean value [4]; (ii) Through a simple nonextensive generalization of the Maxwell ansatz, which is based on the isotropy of the velocity space [5]; (iii) Using a more rigorous treatment based on the nonextensive formulation of the Boltzmann H-theorem [6].

In the astrophysical context, the nonextensive equilibrium velocity distribution related to Tsallis‘ statistics has successfully been applied to stellar polytropes [7], as well as to the peculiar velocity function of galaxies clusters [8]. Some metastable states in pure electron plasmas, and the dispersion relations for electrostatic plane-wave propagation in a collisionless thermal plasma (including undamped Bohm-Gross and Landau damped waves) have also been studied and compared to the standard results [10,11]. In particular, Liu *et al* [12] also showed a reasonable indication for the non-Maxwellian velocity distribution from plasma experiments. All these empirical evidences deal, directly or indirectly, with the q-distribution of velocities for a massive nonrelativistic gas.

In this *letter*, a new application for gravitational systems is discussed with basis on the kinetic approach. In what follows we quantify to what extent the nonextensive effects modify the gravitational instability criterion established by Jeans. More precisely, for this enlarged framework we deduce a new analytical expression for the Jeans dispersion relation which gives rise to an extended instability criterion in accordance to Tsallis statistics.

In his pioneer work, Jeans discussed the conditions under which a fluid (in a static background) becomes gravitationally unstable under the action of its own gravity [9]. Actually, for all scales in the Universe (stars, galaxies and clusters), such an instability is the key mechanism for explaining the gravitational formation of structures. The noteworthy conclusion is that perturbations with mass greather than a critical value M_J (Jeans' mass) may grow thereby producing gravitationally bound structures, whereas perturbations with a mass smaller than M_J do not grow and behave like acoustic waves. This gravitational instability criterion remains basically valid and plays a fundamental role even in the expanding Universe [13,14]. In terms of the wavelengths of a fluctuation, Jeans criterion says that λ should be greater than a critical value $\lambda_J = \sqrt{\pi v_s^2/(G\rho_0)}$, which is known as the *Jeans wavelength*. In this formula G is the gravitational constant, ρ_0 is the unperturbed matter density and v_s is the sound speed for adiabatic perturbations. The same criterion holds for a collisionless self-gravitating cloud of gas, except that the sound speed v_s is replaced by the velocity dispersion σ [13]. In what follows, we focus our attention on the kinetic description of a many particles system within the nonrelativistic gravitational context.

Let us now consider an infinite self-gravitating collisionless gas described by a distribution function $f(\mathbf{v}, \mathbf{r}, t)$ slightly departed from equilibrium. If $f_0(v)$ corresponds to the unperturbed homogeneous and time-independent equilibrium distribution, sometimes referred to as "Jeans Swindle", the resulting particle distribution function may be approximated as

$$f = f_0(v) + f_1(\mathbf{v}, \mathbf{r}, t) \quad , \quad f_1 \ll f_0 \quad , \tag{3}$$

where f_1 is the corresponding perturbation in the distribution function. Following standard lines, the dynamic behavior of this system can be described by the linearized Vlasov and Poisson equations. By neglecting up to second-order terms in the expansion of the distribution function one obtains

$$\frac{\partial f_1}{\partial t} + \mathbf{v}.\frac{\partial f_1}{\partial \mathbf{r}} = \nabla\phi_0.\frac{\partial f_1}{\partial \mathbf{v}} + \nabla\phi_1.\frac{\partial f_0}{\partial \mathbf{v}} \quad , \tag{4}$$

$$\nabla^2\phi_1 = 4\pi G\int f_1(\mathbf{r}, \mathbf{v}, t)d^3v \quad , \tag{5}$$

where $\phi_0(\mathbf{r})$ and $\phi_1(\mathbf{r})$ are, respectively, the unperturbed gravitational potential and its first order correction. As is well known, if "Jeans Swindle" is assumed, one may set the unperturbed potential $\phi_0 = 0$. In this case, the solutions of the above equations can be written as $f_1(\mathbf{r}, \mathbf{v}, t) = F(\mathbf{v}) \exp\{i(\mathbf{k}\cdot\mathbf{r} - \omega t)\}$ and $\phi_1(\mathbf{r}, t) = \phi \exp\{i(\mathbf{k}\cdot\mathbf{r} - \omega t)\}$ provided that $F(\mathbf{v})$ and ϕ satisfy the relations

$$(\mathbf{k}\cdot\mathbf{v} - \omega)F(\mathbf{v}) - \phi\mathbf{k}.\frac{\partial f_0}{\partial \mathbf{v}} = 0 \quad , \tag{6}$$

$$k^2\phi = -4\pi G \int F(\mathbf{v}) d^3v \quad . \tag{7}$$

Combining these expressions one obtains the dispersion relation between ω and $\mathbf{k}$

$$1 + \frac{4\pi G}{k^2} \int \frac{\mathbf{k} \cdot \partial f_0/\partial \mathbf{v}}{\mathbf{k} \cdot \mathbf{v} - \omega} d^3v = 0 \quad . \tag{8}$$

The standard instability analysis follows from the above dispersion relation when one takes $f_0(\mathbf{v})$ as the Maxwellian velocity distribution.

Consider now the q-nonextensive framework proposed by Tsallis. In this case, the equilibrium distribution function $f_0(\mathbf{v})$ can be written as [5,11]

$$f_0(\mathbf{v}) = \frac{\rho_0 B_q}{(2\pi\sigma^2)^{3/2}} \left[1 - (q-1)\frac{v^2}{2\sigma^2}\right]^{\frac{1}{q-1}} \quad , \tag{9}$$

where the normalization constant reads

$$B_q = \frac{(3q-1)(q+1)}{4} \frac{\sqrt{q-1}\,\Gamma(\frac{1}{q-1}+\frac{1}{2})}{\Gamma(\frac{1}{q-1})} \tag{10}$$

for $q \geq 1$, and

$$B_q = \left(\frac{3q-1}{2}\right) \frac{\sqrt{1-q}\,\Gamma\left(\frac{1}{1-q}\right)}{\Gamma\left(\frac{1}{1-q}-\frac{1}{2}\right)} \tag{11}$$

for $\frac{1}{3} < q < 1$. Here σ is the velocity dispersion and ρ_0 is the equilibrium density. As one may check, for $q < 1/3$, the q-distribution (9) is unormalizable, while for $q > 1$, it exhibits a thermal cutoff on the maximum value allowed for the velocity of the particles, which is given by

$$v_{max} = \sqrt{\frac{2\sigma^2}{q-1}} \quad . \tag{12}$$

This thermal cut-off is absent when $q \leq 1$, that is, v_{max} is also unbounded for these values of the q-parameter. In this connection, it is worth notice that the spirit of the H-theorem is totally preserved for this nonextensive velocity distribution. As a matter of fact, by introducing a generalized collisional term, $C_q(f)$, it has been shown that the entropy source is definite positive for $q > 0$, and does not vanish unless the q-equilibrium distribution function assumes the above power-law form [6]. In addition, taking into account that $\lim_{|z|\to\infty}\Gamma(z+a)/[z^a\Gamma(z)] = 1$, as well as that $\lim_{q\to 1}[1-(q-1)x^2]^{1/(q-1)} = \exp(-x^2)$ [15], the expressions (10) and (11) defining B_q reduce to $B_1 = 1$, and as should be expected the distribution function f_0 reduces to the exponential Maxwellian distribution

$$f_0(\mathbf{v}) = \frac{\rho_0}{(2\pi\sigma^2)^{3/2}}\exp(-\frac{v^2}{2\sigma^2}) \quad . \tag{13}$$

In FIG. 1 we have plotted the nonextensive distribution as a function of $v^2/2\sigma^2$ for some selected values of the q parameter. As can be seen, for $q > 1$ the distribution exihibts a cutoff on the maximum value allowed for the velocity of the particles. In the limit $q \to 1$ we see from (12) that $v_{max} \to \infty$ and the power-law behavior reduces to the exponential Maxwellian distribution. For $q < 1$ the cutoff is also absent with the curves decreasing less rapidly than in the Maxwellian case. Note also the convergence from power-law to the standard Gaussian curve as long as the q parameter approaches unity.

Now, substituting the nonextensive equilibrium distribution given by (9) into the dispersion relation (8) one finds

$$1 - \frac{4\pi G\rho_0}{\sigma^5}\frac{B_q}{(2\pi)^{3/2}}\int \frac{v_x\left[1-(q-1)\frac{v^2}{2\sigma^2}\right]^{\frac{2-q}{q-1}}}{kv_x - \omega}d^3v = 0 \tag{14}$$

where the v_x axis has been chosen in the direction of $\mathbf{k}$.

In the study of Jeans instability, the boundary between stable and unstable solutions is achieved by setting $\omega = 0$ in (14). For this value of ω the above integral is can easily evaluated and the result is a q-parameterized family of critical wavenumbers k_q given by

$${k_q}^2 = {k_J}^2\frac{(3q-1)}{2} \tag{15}$$

where ${k_J}^2 = 4\pi G\rho_0/\sigma^2$ is the classical *Jeans wavenumber* [13]. From (15) we have the q-wavelenght $\lambda_q = 2\pi/k_q = \lambda_J\sqrt{2/(3q-1)}$ where λ_J is the *Jeans wavelenght*. Note that the classical value of the Jeans wavenumber and wavelenght as obtained from fluid theory are recovered only if $q = 1$. For $q > 1$ ($1/3 < q < 1$) the q-wavelenght λ_q is decreased (increased) by the factor $\sqrt{2/(3q-1)}$. Associated to λ_q there is a critical mass M_q, defined as the mass contained within a sphere of diameter λ_q, which is given by $M_q = \frac{4\pi}{3}\rho_0(\lambda_q/2)^3 = M_J[2/(3q-1)]^{3/2}$, where M_J is the critical *Jeans mass*. Similarly, we see that for $q > 1$ ($1/3 < q < 1$) the critical mass is decreased (increased) by the factor $[2/(3q-1)]^{3/2}$. Therefore, in comparison with a Maxwellian gas ($q = 1$), a nonextensive collisionless self-gravitating system is more (or less) unstable depending on the value assumed by the Tsallis q-parameter.

Let us now discuss in more detail the unstable modes. As happens in the perturbed fluid theory, fluctuations with wavelenghts $\lambda > \lambda_J$ will be unstable. In order to check what happens in the present kinetic approach we set $\omega = i\gamma$, where γ is real positive, and insert this into the dispersion relation (14). As before, choosing the v_x axis in the direction of $\mathbf{k}$, the following dispersion relation is obtained

$$\frac{k^2}{{k_q}^2} = 1 - \sqrt{\pi}\beta^2 I_q(\beta) \; , \tag{16}$$

where $\beta = \gamma/\sqrt{2}k\sigma$ and $I_q(\beta)$ is a q-dependent integral given by

$$I_q(\beta) = \frac{\sqrt{q-1}\Gamma(\frac{1}{q-1}+\frac{1}{2})}{\Gamma(\frac{1}{q-1})}\frac{q+1}{\pi}\int_0^{\frac{1}{\sqrt{q-1}}}\frac{[1-(q-1)x^2]^{\frac{1}{q-1}}}{\beta^2+x^2}dx \tag{17}$$

for $q > 1$ and

$$I_q(\beta) = \frac{\sqrt{1-q}\Gamma(\frac{1}{1-q})}{\Gamma(\frac{1}{1-q}-\frac{1}{2})}\frac{2}{\pi}\int_0^{\infty}\frac{[1-(q-1)x^2]^{\frac{1}{q-1}}}{\beta^2+x^2}dx \tag{18}$$

for $1/3 < q < 1$. It should be noticed that the limit $q \rightarrow 1$ of both expressions is

$$\lim_{q\rightarrow 1} I_q(\beta) = \frac{2}{\pi}\int_0^{\infty}\frac{e^{-x^2}}{\beta^2+x^2}dx = \frac{e^{\beta^2}}{\beta}[1-\mathbf{erf}(\beta)] \quad , \tag{19}$$

with the expression (16) reducing to

$$\frac{k^2}{{k_J}^2} = 1 - \sqrt{\pi}\beta e^{\beta^2}[1-\mathbf{erf}(\beta)] \quad , \tag{20}$$

which is the standard dispersion relation for stellar systems [13]. For an arbitrary value of q, the dispersion relation is given explicitly by the integral form (16) which can be numerically evaluated. In Fig. 2 we plot the unstable branches for several values of q along with the dispersion relation for an infinite homogeneous fluid which is given by the solid straight line (see [13] for comparison).

As can be seen from these plots, for $q > 1$ the system presents instability even for wavenumbers of the disturbance greater than the standard critical Jeans value (k_J). In other words, this kind of q-gaseous system is more unstable than the one described by the standard Maxwellian curve, and therefore, structures are more easily formed in this nonextensive context. However, for $q < 1$ the system may remain stable even for disturbance with wavenumber smaller than the Jeans wavenumber. This yields a generalization of Jeans gravitational instability mechanism within Tsallis' nonextensive thermostatistical context. The standard Jeans criterion is thus modified with critical values of mass and lenght depending explicitly on the nonextensive q-parameter.

In Fig. 3 new plots of the dispersion relation (unstable branches) are presented. Different from FIG.2, instead of the ratio involving the critical Jeans wavenumber (k_J), the behavior of the unstable branches are analyzed as a function of the ratio k/k_q.

As should be expected at first sight, the curves are convergent for the critical ratio ($k/k_q = 1$), but the growth rates are slightly different for each value of q. Note also that for all values of q, the results are quite different of what happens in the macroscopic fluid approach for gravitational instability (solid straight line).

Summarizing, we have investigated a generalization of the Jeans gravitational instability along the lines of the nonextensive statistical formalism proposed by Tsallis. The main interest of our results rests on the fact that the Maxwellian curve may provide only a very crude description of the velocity distribution for a selfgravitating gas, or more generally for any system endowed with long range interactions. As we have seen, a well determined criterion for gravitational instability is not a privilege of the exponential velocity distribution function, but is shared by an intire power-law family of functions (named q-exponentials) which includes the standard Jeans result for the Maxwellian distribution as a limiting case ($q = 1$). In general grounds, the basic instability criterion is maintained: perturbations with $k > k_q$ do not grow (or are damped even considering that the collisionless q-gas is a time reversible system), while instability takes place for $k < k_q$. However, unlike Jeans' treatment, in this context there exist a family of growth rates parametrized by the nonextensive q-parameter. In particular, for $q > 1$ (power-law distributions with cutoff) the system presents instability even for wavenumbers of the disturbance greater than the standard critical Jeans value k_J. Finally, since the gravitational instability Jeans' criterion remains basically true for an expanding Universe [14], this surprizing result suggest that nonextensive effects may have important consequences to the galaxy formation process.

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Figure Captions

1. Nonextensive velocity distribution function $f_0(\mathbf{v})$ for typical values of the q parameter. Power law distributions with $q > 1$ exhibts a thermal cutoff on the maximum value allowed for the velocity of the particles. This cutoff is absent for $\frac{1}{3} < q \leq 1$, and the distribution is unnormalizable for $q < \frac{1}{3}$.

2. Unstable branches of the dispersion relations for an infinite self-gravitating collisionless gas obeying the nonextensive Tsallis q-statistics. For comparison the straight line representing the dispersion relation for an infinite homogeneous fluid has also been plotted. We see that for $q > 1$ (power-law distributions with cutoff), the system is unstable even for wavelenght of the disturbance smaller than the standard Jeans lenght value (λ_J).

3. Unstable branches of the dispersion relations as a function of the critical wavenumbers k_q. Notice that the magnitude of k_q is related to the critical Jeans wavenumber by equation (15). As in Fig.2, the straight solid line stands to the case of an infinite homogeneous fluid.

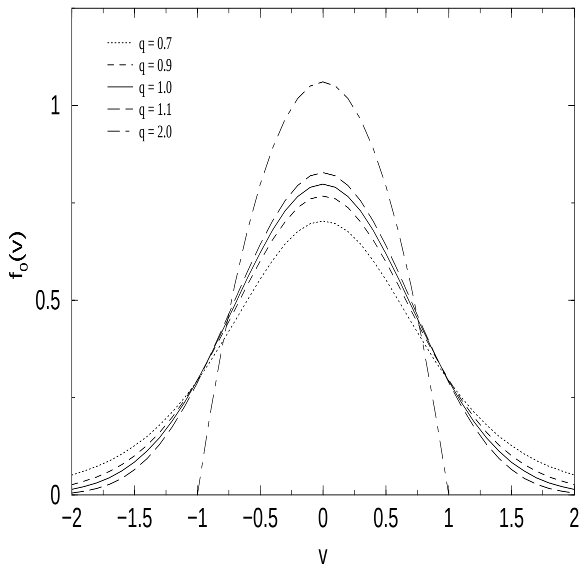
q = 0.7
q = 0.9
q = 1.0
q = 1.1
q = 2.0
1
0.5
0
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
$f_o(v)$
v

FIG. 1.

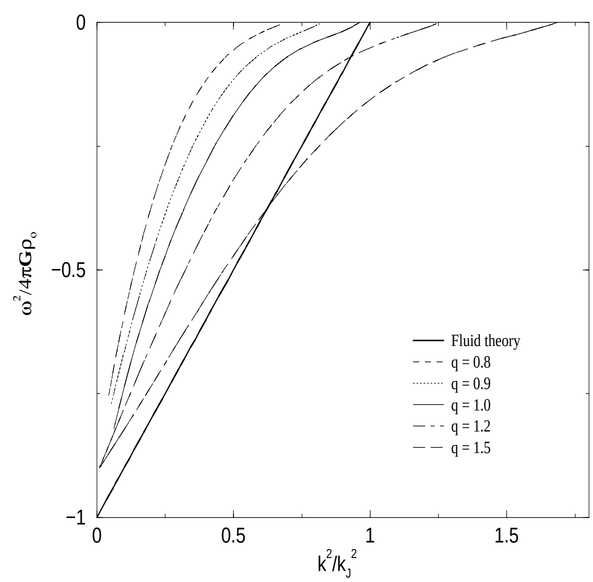
Fluid theory
q = 0.8
q = 0.9
q = 1.0
q = 1.2
q = 1.5
0
−0.5
−1
0
0.5
1
1.5
$\omega^2/4\pi G\rho_o$
k^2/k_J^2

FIG. 2.

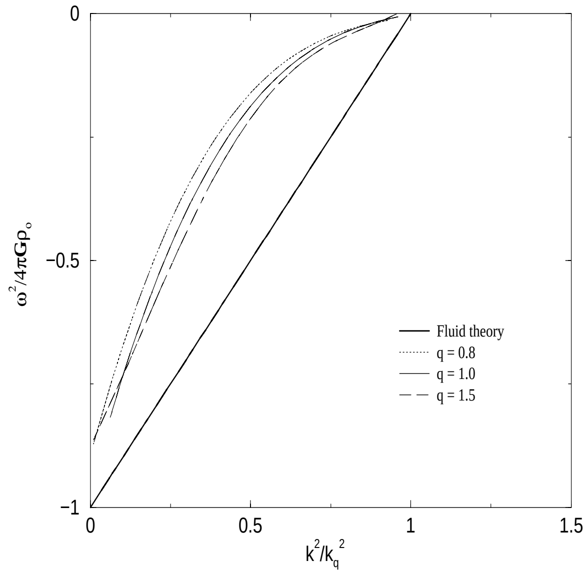

FIG. 3.